Inequalities-I.

Problem 1. Prove the general rearrangement inequality: if $x_1 \geqslant \ldots \geqslant x_n$, $y_1 \geqslant \ldots \geqslant y_n$, then for any permutation $i_1, \ldots, i_n$ of $1, \ldots, n$:

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n \geqslant x_1y_{i_1} + x_2y_{i_2} + \cdots + x_ny_{i_n}.$$

Problem 2. The numbers $1, 2, \ldots, 9$ are written on the board. Consider all consecutive triples of digits and sum the corresponding three-digit numbers. What is the maximal possible value of the sum?

Problem 3. Let $a, b > 0$. Prove, that $(a+b)^n \leqslant 2^{n-1}(a^n + b^n)$

Problem 4. Let $a, b, c > 0$. Prove, that

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geqslant \frac{3}{2}.$$

Problem 5. The numbers 1 to 100 are written on a 10×10 board (1-10 in the first row, etc). We are allowed to pick any number and two of its neighbors (horizontally, vertically, or diagonally — but our choice must be consistent), increase the number with 2 and decrease the neighbors by 1, or decrease the number by 2 and increase the neighbors by 1. At some later time the numbers in the table are again $1, 2, \ldots, 100$. Prove that they are in the original order.

Problem 6. Let ABC be a triangle, a, b, c — lengths of its sides and h_a, h_b, h_c the corresponding altitudes. Prove the following bound on the area S of ABC:

$$S \leqslant \frac{(a+b+c)(h_a+h_b+h_c)}{6}.$$

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